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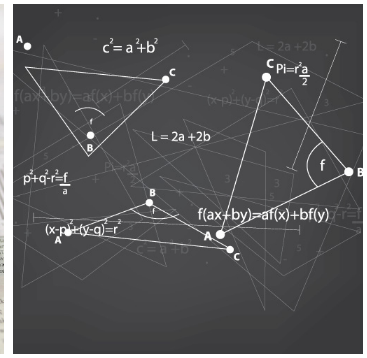

Fibonacci Numbers

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Has anyone not heard of Fibonacci numbers? They're found in nature, literature, movies, and well, they're famous. They're also on the Internet, so if you really want to delve into the subject, just go online. There, I imagine, you'll get the official version. In this article, you'll get mine. Some resemblance should be expected and would not be coincidental – after-all, all the characters "living or dead" are all dead. Dead since the 11th Century. So where did these numbers come from? Answer: rabbits; pairs of rabbits.

Fibonacci was tackling the problem of rabbit propagation. I can't honestly say for sure whether he was interested in the problem ecologically or as a math-puzzle. Sources I've read are conflicting and Fibonacci possessed a middle-age mind – chronologically and historically. Regardless, here's how we'll proceed: (1) Rabbits go from baby pairs to adult pairs in one month, (2) an adult pair can conceive a baby pair after one month of adulthood (3) Rabbits live forever. All reasonable assumptions!

Rabbit Pairs	Jan	Feb	March	April	May
Adults	0	1	1	2	3
Babies	1	0	1	1	2
Total	1	1	2	3	5

The pattern that emerges is that of any three consecutive numbers, the last is the sum of the prior two. Or, speaking mathematically: $F_{n+1} = F_{n-1} + F_n$. The first 35 Fibonacci number are listed below:

Fibonacci Numbers

F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	F_{11}	F_{12}
1	1	2	3	5	8	13	21	34	55	89	144

F_{13}	F_{14}	F_{15}	F_{16}	F_{17}	F_{18}	F_{19}	F_{20}	F_{21}	F_{22}	F_{23}	F_{24}
233	377	610	987	1597	2584	4181	6765	10946	17711	28657	46368

F_{25}	F_{26}	F_{27}	F_{28}	F_{29}	F_{30}	F_{31}	F_{32}	.... F_{35}
75025	121393	196418	317811	514229	832040	1346269	2178309	9227465

What makes these numbers so intriguing to mathematicians are the amazing patterns they possess. I have my favorites. The first can be Googled, but the proof is my own, the second I discovered myself, and the third – the most magical one, is my twist on well-known results.

The first pattern is known as Cassini's Identity. Look at any three consecutive Fibonacci numbers, for example, 13, 21 and 34. Square the middle one ($21^2 = 441$) then multiply the outer two by each other ($13 \times 34 = 442$). That 442 and 441 differ by one is no chance result – it always is the case. Try any three pair yourself.

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Fibonacci Numbers continued...

The second pattern has no formal name but is just as interesting. Identify the position of any Fibonacci number, say $F_7 = 13$, then look at the Fibonacci number in the doubled position – in this case $F_{14} = 377$. Turns-out that 13 divides 377. (377 = 13 x 29). Not only is this true for any doubled-position, it is true for any position multiple. That is, $F_{21} = 10946 = 13x\ 842$ and $F_{28} = 317811 = 13 \times 24447$. Put another way, if a number divides the position of a Fibonacci number then the corresponding Fibonacci numbers divide as well. Let's look at $F_{35} = 9227465$. As 5 and 7 divide 35 then 5 ($F_5 = 5$) and 13 should divide 9,227,465 and sure enough they do, 9,227,465 = 5 x 13 x 141,961

But we've saved the best for last. Recall the Golden Ratio: the value of x where $x/1 = (x + 1)/x$. We talked about this in a previous newsletter where we found x to be 1.61803… Now pick two consecutive Fibonacci numbers, here big is better. Say F_{31} and F_{32}, 1,346,269 and 2,178,309. Divide the former into the later: 2,178,309 ÷ 1,346,269 = 1.61803…. Now if that's not pulling a rabbit out of a hat, what is?

We wrap things up with a proof of our result. No harm in skipping if you've read enough already

Here goes:

$$
\begin{aligned}
&F_n \\
&F_{n+1} = F_n + F_{n-1} \\
&F_{n+2} = (F_{n-1} + F_n) + F_n = F_{n-1} + 2F_n \\
&F_{n+3} = (F_{n-1} + 2F_n) + (F_{n-1} + F_n) = 2F_{n-1} + 3F_n
\end{aligned}
$$

More generally, $F_{n+K} = F_k F_{n-1} + F_{k+1} F_n$

If k = n, we have $F_{2n} = F_n (F_{n-1} + F_{n+1})$, proving that F_n divides F_{2n}

For $F_{(a+1)d} = F_{d+ad} = F_{ad} F_{d-1} + F_{ad+1} F_d$ -> if F_d divides F_{ad} then it divides $F_{(a+1)d}$

As F_d divides F_d, by mathematical induction we have our proof.

Going in reverse:

$$
\begin{aligned}
&F_{n-2} = F_n - F_{n-1} \\
&F_{n-3} = F_{n-1} - F_{n-2} = F_{n-1} - (F_n - F_{n-1}) = 2F_{n-1} - F_n \\
&F_{n-4} = F_{n-2} - F_{n-3} = (F_n - F_{n-1}) - (2F_{n-1} - F_n) = 2F_n - 3F_{n-1}
\end{aligned}
$$

In the event k is even: $F_{n-k} = F_n F_{k-1} - F_k F_{n-1}$

And in the event k is odd: $F_{n-k} = F_k F_{n-1} - F_n F_{k-1}$

Hence if k = n – 1 ->$1 = F_1 = F_n F_{n-2} +/- F_{n-1} F_{n-1}$ giving us Cassini's formula.

Our last result follows from the observation that $F_{n+1}/F_n = (F_n + F_{n-1})/F_n = 1 + (F_{n-1}/F_n) = 1 + 1/(F_n/F_{n-1})$.

Letting n go to infinity and assuming convergence (the proof of which is out of our league), we let x represent the limiting value giving us: $x = 1 + 1/x = (x + 1)/x$, the defining equation of the Golden Ratio.